

The Glassy Potts Model

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We introduce a Potts model with quenched, frustrated disorder, that enjoys of a gauge symmetry that forbids spontaneous magnetization, and allows the glassy phase to extend from T_c down to $T = 0$. We study numerical the 4 dimensional model with $q = 4$ states. We show the existence of a glassy phase, and we characterize it by studying the probability distributions of an order parameter, the binder cumulant and the divergence of the overlap susceptibility. We show that the dynamical behavior of the system is characterized by aging.

The generalization of the Ising model to a frustrated model containing quenched disorder, the Ising spin glass, has provided us with a large amount of new physics [1]. Replica Symmetry Breaking has been found in the Mean Field theory [2], and mainly numerical simulations strongly hint to its validity in finite dimensional disordered Ising spin glasses [3].

The need of a generalization of such systems to the Potts models has been clear very soon: technical motivations are obvious, while physical motivations include the need of describing systems where the Z_2 symmetry of the Ising model is not relevant (real glasses being potentially among them [4]). The most straightforward construction of a Potts spin glass, where the spin variables can be in q states and are randomly connected by a positive or a negative coupling, has been analyzed in detail [5]- [13], but it has the unappealing feature (that we will justify in the following) of acquiring a spontaneous magnetization (i.e. of entering a phase with usual ferromagnetic ordering) at low T values. The glassy regime is only present in a small T region, making it unpractical to be studied numerically and unpalatable for a faithful description of real glasses (that do not order at low temperatures).

Here we will define and study what we consider to be the “naturally glassy” generalization of the Ising spin glasses to the Potts model. We will study the finite dimensional version of the model, and we will show that these systems do indeed undergo a phase transition that leads them to a glassy phase, different from the usual Sherrington Kirkpatrick spin glass phase.

We regard here as crucial the exact *gauge invariance* that is found in the usual Ising spin glasses (both in

finite number of dimensions and in the mean field approximation). The Hamiltonian of the Ising spin glass is $-\sum \sigma_i J_{i,j} \sigma_j$, where the sum runs over couples of first neighboring sites of the (simple hyper-cubic) d dimensional lattice, the J are ± 1 with uniform probability (or Gaussian variables), and the spin σ take the values ± 1 . Let us consider the site i and transform $\sigma_i \rightarrow -\sigma_i$. If at the same time we flip all the $2d$ couplings $J_{i,j}$ involving the site i (and one of its first neighboring sites j) the energy of the system does not change. Because of this symmetry the expectation value of the magnetization is zero, and a ferromagnetic phase is not allowed.

The generalization to a disordered model of [5]- [13] is not protected by such a local symmetry, and is allowed to magnetize (as it indeed does). We propose instead a generalization to a quenched, frustrated spin model, where the gauge invariance is preserved. The Hamiltonian of our model is

$$H \equiv - \sum_{\langle i,j \rangle} \delta_{\sigma_i, \Pi_{i,j}(\sigma_j)} , \quad (1)$$

where the sum runs over first neighboring sites on a simple cubic hyper-lattice (or over all site couples in the mean field model), the spin variables σ can take q values $(0, 1, \dots, q-1)$, and the $\Pi_{i,j}$ are link attached, quenched, random permutations of $(0, \dots, q-1)$ (there are $q!$ of them). In the following we will denote this model by $\mathcal{M}_{q!}$. It is clear that, when written as a sum of delta functions, the Ising spin glass has exactly this form: in this sense this is a very natural generalization of the model, where we just increase the number of allowed states. The link $\langle i, j \rangle$ will give a non-zero contribution to the action not, as in the usual Potts model, if $\sigma_i = \sigma_j$, but

if σ_i is equal to $\Pi_{i,j}(\sigma_j)$. In this model the same gauge symmetry we have described before protects us against magnetizing: if we transform σ_i from 0 to 1 we will interchange in the quenched random permutations involving the site i the state 0 and the state 1. This feature makes this model (the glassy Potts model) a good candidate to the description of the glassy state.

The model we have just defined, $\mathcal{M}_{q!}$, has a drawback: it is very difficult to check if it has reached thermal equilibrium. The Z_2 symmetry of the Ising model is indeed precious at this end: in a spin glass checking the symmetry of the probability distribution of the overlap is crucial for establishing thermalization [3]. Our first numerical simulations of the model $\mathcal{M}_{q!}$ have confirmed how difficult it is to establish on firm grounds thermalization without being able to count on a slow mode that has to exhibit a symmetry. It is possible to solve this problem at least for even values of the number of allowed states, q . One considers a permutation R such that $R^2 = 1$ (for example we can change the state $2k$ with the state $2k + 1$ for $k = 0, \dots, \frac{q}{2} - 1$), and allows in the δ function only permutations that commute with R . We have introduced the model \mathcal{M}_{co} (where *co* stands for commutative) where the Hamiltonian of equation (1) contains only permutation Π that commute with the permutation $R = (0, 1, 2, 3) \rightarrow (2, 3, 0, 1)$, i.e. $\Pi R = R \Pi$. This model is symmetric under R , and invariance under R can be tested in order to check if thermal equilibrium has been reached. In order to do that we have defined a modified overlap ω , that is one if two spins are in the same state, -1 for the couples $(0, 2)$, $(1, 3)$, $(2, 0)$ and $(3, 1)$, and zero otherwise (q will be the usual overlap, where we sum one if two spin are equals and zero otherwise). Because of the symmetry we have introduced by selecting only R commuting permutations Π the probability distribution $P(\omega)$ is symmetric at equilibrium under $\omega \rightarrow -\omega$. The two models are expected and turn out to be equivalent, as we will show in the following. When using \mathcal{M}_{co} it is easy to check thermalization, and the coincidence of the results with the ones obtained when studying $\mathcal{M}_{q!}$ shed light on their physical meaning.

We have studied $\mathcal{M}_{q!}$ and \mathcal{M}_{co} with $q = 4$ states in 4 spatial dimensions d . We have used a normal Monte Carlo method. We will present thermalized data in the broken phase for lattices of volume $L^4 = 4^4$ and 5^4 , and data in the warm phase for a 8^4 lattice. The data on the two smaller lattice volumes have been obtained from a slow annealing with ten million full sweeps of the lattice at each temperature point, the one with $L = 8$ have used one million sweeps per T point. We have averaged over ten realizations of the disorder. We have preferred to have long thermal runs, since thermalization of the samples needs to be completely sure to make the results reliable, and to keep the number of samples quite small. The numerical simulations have taken of the order of two months of medium size workstations. We only report re-

sults for which we are sure of having reached full thermalization (the main criterion used being the symmetry of the probability distribution of \mathcal{M}_{co} , and the request of a good stability in time of the observables).

Working on lattices of linear size 4 and 5 we have succeeded to get some control over the finite size behavior. A large amount of evidence, that we will describe in the following, makes clear the existence of a phase transition to a low T glassy phase.

The Binder parameters of the modified overlap

$$g_\omega(T) \equiv \frac{1}{2} \left(3 - \frac{\overline{\langle \omega^4 \rangle}}{\overline{\langle \omega^2 \rangle}^2} \right) \quad (2)$$

shows the clear signature of a phase transition. For \mathcal{M}_{co} $g_\omega \simeq 0$ with good accuracy in the warm phase (because of the symmetry we have implemented). It becomes different from zero at $T \simeq 1.4$, and grows basically linearly (in our statistical accuracy) at low T . At $T = 1.2$ (the lowest T value we are sure we have thermalized both at $L = 4$ and $L = 5$) $g_\omega \simeq \frac{1}{2}$. In the precision given by our statistical errors (of the average over disorder), evaluated directly for the Binder parameter by a jack-knife method, the $L = 4$ and $L = 5$ results coincide (at $T \simeq 1.2$ the relative error on the Binder parameter is of the order of ten percent: for example for $L = 5$ we have $g_\omega = 0.45 \pm 0.10$). For the model $\mathcal{M}_{q!}$ g_ω has the same pattern but it becomes slightly negative at $T \simeq 1.7$ (but close to zero): at low T it increases like for the other model. The Binder parameter of the straightforward overlap does not behave in an interesting way for both models. From the analysis of the Binder parameters we deduce as a first guess that $T_c \simeq 1.5$.

In figure (1) we plot the Replica Symmetry Breaking parameter introduced in [14],

$$\rho_\omega \equiv \frac{\overline{\langle \omega^2 \rangle}^2 - \overline{\langle \omega^2 \rangle^2}}{\overline{\langle \omega^4 \rangle} - \overline{\langle \omega^2 \rangle}^2}, \quad (3)$$

for \mathcal{M}_{co} (where we have assumed that $\langle \omega \rangle = 0$). ρ seems to give an even clearer signature of the phase transition than the Binder parameter (qualifying it without ambiguities as a Replica Symmetry Breaking transition). In the infinite volume limit ρ is zero if the overlap distribution is self-averaging, and becomes non-zero if Broken Replica Symmetry makes it non-self-averaging. Reference [14] shows that while the Binder parameter is very effective in detecting symmetry breaking accompanied by breaking of spin reversal symmetry, but when spin reversal symmetry is absent ρ tends to be a better estimator. Figure (1) shows a sharp transition (notice that the vertical scale is logarithmic). Again a good estimate for T_c is 1.5. The evidence for a phase transition is clear. The fact we can detect it by using ρ makes clear it is a Replica Symmetry Breaking transition. Again, the situation in

\mathcal{M}_{q_l} is very similar: there is a change of regime close to $T = 1.5$, where ρ does not go to zero with increasing L .

Maybe the most interesting evidence about the behavior of the system comes from the analysis of the probability distributions of the overlap averaged over the different disorder samples. At high T $P(\omega)$ in \mathcal{M}_{q_l} tends to a Gaussian when increasing the lattice size. We show in figure (2) $P(\omega)$ at $T = 1.25$ for $L = 4$ and 5. There is a clear non-trivial behavior (notice that the support in ω is very extended, i.e. $P(\omega)$ is very flat). On larger lattices a peak in $\omega \simeq 0$ is emerging, separated by a flat minimum from a second peak (the minimum becomes sharper at lower T values, but there we know we did not reach full thermal equilibrium and we cannot safely attach a firm significance to the data): this behavior is reminiscent of the one of REM models, and fits with what we expect for a glassy state [4]. The important point here is that we have been able to thermalize the $L = 5$ system even in the cold phase ($T = 1.25$ is the lowest temperature value where we are sure about an adequate thermalization).

It is useful (and necessary) to look at the individual, single sample probability distributions $P_J(\omega)$ in order to qualify the behavior of the individual systems. In figure (3) we plot $N_J(\omega)$ (the non-normalized $P_J(\omega)$) for four typical samples, versus ω at $T = 1.25$. The level of the asymmetry of the histograms is a measure of our statistical error (and the fact the functions look symmetric a sign of a good thermalization). The $N_J(\omega)$ are non-trivial: some samples have double peaks, some have their support close to $\omega = 0$, some have support at zero overlap and peaks at finite overlap. On a qualitative level we remark that system looks *harder* than the usual spin glass: the dynamics is more jumpy, visiting in a quite discontinuous manner different parts of the phase space (that is why checking thermalization has been difficult and crucial). Our evidence seems to suggest that the free energy phase space is golf course like: deep minima do not have large basins of attraction.

The probability distribution for the modified overlap ω in the model \mathcal{M}_{q_l} is similar to the one we have shown. The detailed shape is not exactly the same, but it also becomes non-trivial at $T = 1.25$. On the $L = 5$ lattice a two peak structure starts to emerge from $T = 1.25$ down (one in $\omega \simeq 0$ and one at a finite ω value). The non-modified overlap distributions for both models enjoy the same main features (even if they are non-symmetric, and, as we have already discussed, the thermalization is better checked by using the symmetry of $P(\omega)$ in \mathcal{M}_{co}): it tends to a Gaussian when increasing L at $T = 1/0.6$, it is non-trivial at $T = 1/0.7$, and it develops a two peak structure at lower T values.

We have used the T data in the warm phase from the large, $L = 8$ lattice to fit the divergence of the overlap susceptibility $\chi_\omega \equiv V\langle\omega^2\rangle$, that $T \rightarrow T_c^+$ behaves as $(T - T_c)^{-\gamma}$. From our data we can only give a preliminary estimate, that puts T_c among 1.4 and 1.5 (compatible

with the value we have deduced from the direct analysis of $P(\omega)$) and the exponent γ among 1.3 and 1.5. This value is different from the one quoted for the Ising 4D spin glass, $\gamma = 2.10$ [15].

The dynamical behavior of the system shows all the typical features of the complex dynamics. In figure (4) we show the aging behavior: the spin-spin correlation function $C(t, t_w)$, depending on the waiting time t_w and on the time t , versus t for different waiting times t_w . The rate of the time decay depends on t_w .

Let us notice at last that the energies of the two models are very similar: on the $L = 5$ lattice they are equal in our statistical precision, while on the $L = 8$ lattice they are of the order of one per one thousand. The two models seem to present the same kind of critical (and off-critical) behavior, and our best guess is that they do belong to the same universality class.

We have introduced a disordered generalization of the Potts model that we regard as a very hopefully candidate to describe the glassy state. Our numerical simulations of the 4 state, 4 dimensional model show clearly the existence of a glassy phase, and they stress large differences with the usual Sherrington Kirkpatrick spin glass phase, exhibiting a more discontinuous behavior, reminiscent of the Random Energy Model one step Replica Symmetry Breaking [1].

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- [1] M. Mezard, G. Parisi and M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
 - [2] G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979); J. Phys. A **13**, 1101 (1980); **13**, 1887 (1980); **13**, L115 (1980); Phys. Rev. Lett. **50**, 1946 (1983).
 - [3] E. Marinari, G. Parisi and J. J. Ruiz-Lorenzo, *Numerical Simulations of Spin Glass Systems*, in *Spin Glasses and Random Fields*, edited by P. Young (World Scientific, Singapore 1997), cond-mat/9701016.
 - [4] See for example G. Parisi, *Slow Dynamics of Glassy Systems*, 1996 Varenna Lectures, cond-mat/9705312.
 - [5] D. Elderfield and D. Sherrington, J. Phys. C **16**, L497 (1983); L971; L1169.
 - [6] A. Erzan and E. J. S. Lage, J. Phys. C **16**, L555 (1983).
 - [7] D. J. Gross, I. Kanter and H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985).
 - [8] T. R. Kirkpatrick and P. G. Wolynes, Phys. Rev. B **36**, 8552 (1987); **38**, 4881 (1988).
 - [9] T. R. Kirkpatrick and D. Thirumalai, Phys. Rev. B **37**, 5342 (1988).
 - [10] G. Cwilich and T. R. Kirkpatrick, J. Phys. A **22**, 4971 (1989); **23**, 5029 (1990).
 - [11] M. Scheucher, J. D. Reger, K. Binder and A. P. Young, Phys. Rev. B **42**, 6881 (1990).

- [12] E. De Santis, G. Parisi and F. Ritort, J. Phys. A **28**, 3025 (1995), cond-mat/941093.
- [13] K. Binder, in *Spin Glasses and Random Fields*, edited by P. Young (World Scientific, Singapore 1997),
- [14] E. Marinari, C. Naitza, G. Parisi, M. Picco, F. Ritort and F. Zuliani, cond-mat/9802309.
- [15] G. Parisi, F. Ricci Tersenghi and J. J. Ruiz-Lorenzo, cond-mat/9606051.

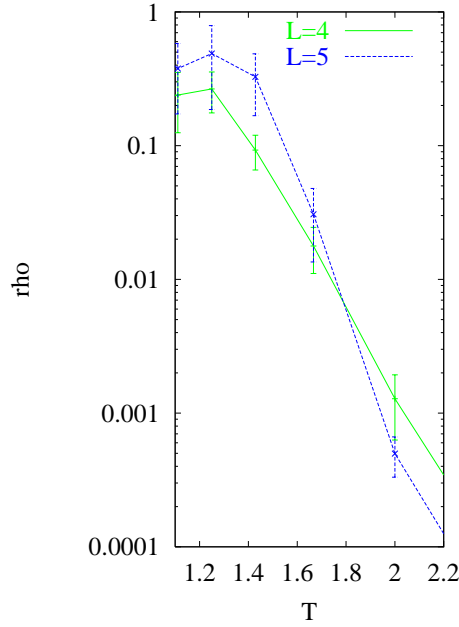


FIG. 1. $\rho(T)$ versus T . Solid line for $L = 4$ and dashed line for $L = 5$. Notice the vertical logarithmic scale.

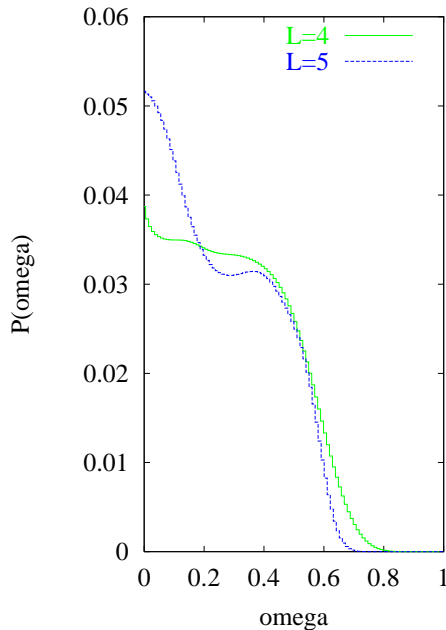


FIG. 2. $P(\omega)$ (symmetrized) versus ω at $T = 1.25$. Annealed runs with ten million sweeps per T point. Solid line for $L = 4$ and dashed line for $L = 5$.

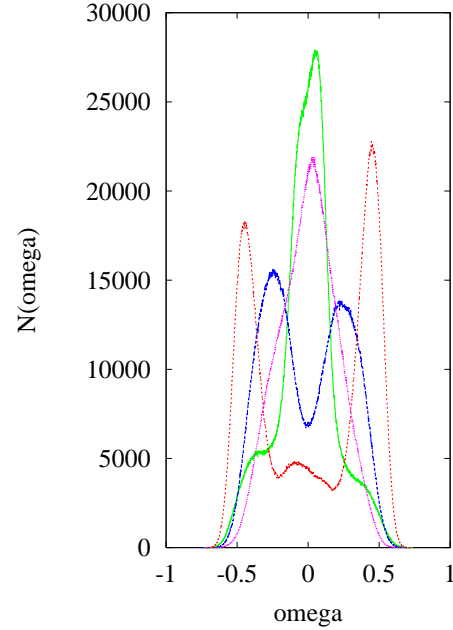


FIG. 3. $N_J(\omega)$ (the non-normalized $P_J(\omega)$) for four typical samples, versus ω at $T = 1.25$. Annealed runs with ten million sweeps per T point.

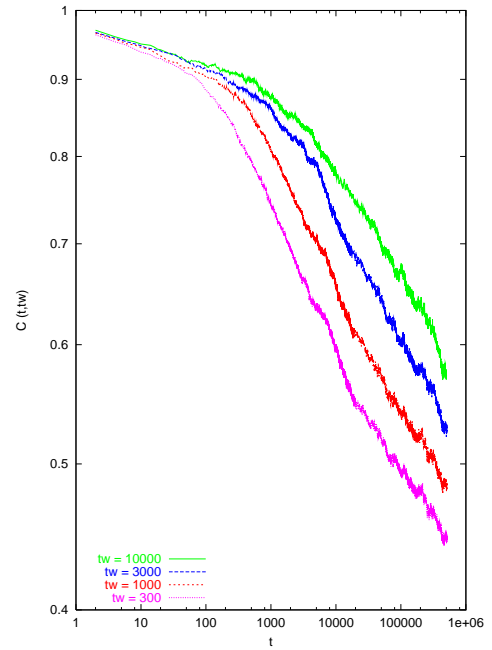


FIG. 4. Dynamical correlation function $C(t, t_w)$ versus t for different values of t_w .